

ARBITRAGE PRICING OF CONVERTIBLE SECURITIES WITH CREDIT RISK

Tomasz R. Bielecki*
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA

Stéphane Crépey†
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France

Monique Jeanblanc‡
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France

Marek Rutkowski§
School of Mathematics
University of New South Wales
Sydney, NSW 2052, Australia
and
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland

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1 Introduction

1.1 General Motivation

It is widely acknowledged (see, for instance, [13, 21, 24]) that a Convertible Bond (CB) has a natural interpretation as a defaultable bond supplemented with an option to exchange this defaultable bond for a given number κ of shares. Thus, convertible bonds are often advertised as products with upside potential and limited downside risk. However, after years of steady growth, the market of convertible bonds has suffered an unprecedented drawback in April–May 2005. Many hedge funds closed their convertible bond positions, while new convertible bond issues became more and more rare. This was largely due to persistently low Credit Default Swap (CDS) spreads and low volatilities that limited the potential benefit of convertible bond arbitrage, and to regulatory changes that made financing by means of convertible bond a less attractive alternative to straight bond financing than before. In addition, some practitioners blamed this crisis on inadequate understanding of the product, that let people think for a while that convertible bonds were a win-win mixture to both issuers and holders, up to the point where disappointment changed their mind the other way around. So, many actors in the equity-to-credit universe closed their positions after the unexpected simultaneous rise in the General Motors CDS spreads and stock price in May 2005 (cf. [25]). Associated with this lack of understanding, deficiency of convertible bond software caused unexpected losses, which hastened sell-off of convertible bonds.

In this paper, we attempt to shed more light on the mathematical modeling of convertible bonds, thus continuing the previous research presented, for instance, in [1, 3, 11, 13, 15, 21, 22, 23, 24]. In particular, we consider the problem of the decomposition of a CB into bond component and option component. This decomposition is indeed well established in the case of an ‘exchange option’, when the conversion can only occur at maturity (see [21]). However, it was not yet studied in the case of a real-life convertible bond.

More generally, we shall consider generic *Convertible Securities* (CS), encompassing convertible bonds as a special case. Since convertible bonds are generally not default-free, we shall examine *defaultable* convertible securities in a general framework of a fairly general market model in which prices of primary assets are assumed to follow semimartingales (see [12] or [15]) and the random moment of default is exogenously given.

1.2 Main Results

The main results of this work are Theorems 3.1 and 5.1. Theorem 3.1 provides a characterization of the set of arbitrage prices of a convertible security in terms of values of the corresponding Dynkin games,¹ in the same line as Kallsen and Kühn [15, Theorem 2.9], but under different assumptions. Specifically, we do not assume inequality (4) below, which is typically not satisfied in the case of defaultable convertible bonds, but we impose instead alternative conditions, that are naturally fulfilled for any defaultable convertible security. Theorem 5.1 furnishes a rigorous decomposition of a convertible security. In particular, it allows one to give a definite meaning (see Definition 5.1) to commonly used terms, such as: the *spread* and the *implied volatility* of a convertible bond.

This paper provides theoretical underpinning for a more extensive research continued in Bielecki et al. [5, 6], where more specific market models are introduced, and more explicit valuation and hedging results are established. In [5], we derive valuation results for a convertible security in the framework of a default risk model based on the hazard process, and we provide a characterization of minimal super-hedging strategies of a convertible security within the default intensity set-up through solutions of doubly reflected backward stochastic differential equations. In [6], we introduce a particular Markovian model of credit risk, and we analyze the related variational inequalities.

¹For general results on Dynkin games, see, for instance, Dynkin [14], Kifer [18], Lepeltier and Maingueneau [20].

1.3 Set-Up

In the sequel, we assume that the evolution of the underlying market can be modeled in terms of stochastic processes, which are defined on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, with the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ satisfying the usual conditions. Here, we denote by \mathbb{P} the statistical (objective) probability measure. We do not specify the underlying market at this stage so that, in particular, it may be incomplete, but we always assume that the market model is free of arbitrage opportunities. We assume that the *discount factor* process (that is, the inverse of the savings account) β_t , $t \in \mathbb{R}_+$, is a \mathbb{G} -adapted, positive, and locally bounded process, with bounded variation and càdlàg sample paths.

Assume that continuously paid dividends are proportional to the market value S_t of the underlying stock, so that the dividend process D satisfies $dD_t = q_t S_t dt$, where $q_t \geq 0$ is the dividend yield on S . Then, by standard no-arbitrage arguments, the discounted, dividend-adjusted stock value $(\beta_t Q_t S_t)_{t \geq 0}$, where $Q_t = e^{\int_0^t q_u du}$, is a \mathbb{G} -martingale under an equivalent martingale measure for the underlying market. In this work, we find it convenient to adopt an abstract convention that the *dividend-adjusted stock value* equals $\widehat{S}_t = Q_t S_t$, where a generic *dividend adjustment factor* Q follows a \mathbb{G} -adapted and locally positively bounded process, with bounded variation and càdlàg sample paths. The initial conditions $Q_0 = 1$ and $\beta_0 = 1$ are also imposed. We assume that the stock price S is a \mathbb{G} -semimartingale with non-negative càdlàg trajectories, and that the discounted, dividend-adjusted stock value βS is a \mathbb{G} -martingale under a martingale measure for the underlying market. Let us stress that we assume throughout that we deal with a perfect market model with an unrestricted trading in the underlying primary assets.

2 Game Options

2.1 Payoffs of a Game Option

As it is well known (see, for instance, Kifer [17]), a convertible bond with no call notice period can be formally seen as a special case of the so-called *game option*, which was introduced in Kifer [17] (see also Kallsen and Kühn [15]).

For any $t \in [0, T]$, we write \mathcal{G}_T^t to denote the set of all \mathbb{G} -stopping times with values in $[t, T]$. Let 0 (respectively T) stand for the *inception date* (respectively the *maturity date*) of a game option.

Definition 2.1 A *game option* is a contract with discounted payoff, as seen from the perspective of the holder, given by

$$\mathcal{J}(\tau_p, \tau_c) = \mathbb{1}_{\{\tau_p \leq \tau_c\}} \beta_{\tau_p} \mathcal{L}_{\tau_p} + \mathbb{1}_{\{\tau_p > \tau_c\}} \beta_{\tau_c} \mathcal{U}_{\tau_c}, \quad (1)$$

where $\tau_p, \tau_c \in \mathcal{G}_T^0$ are stopping times under the control of the holder and the issuer of a game option respectively. The *lower payoff process* $\mathcal{L} = (\mathcal{L}_t)_{t \in [0, T]}$ and the *upper payoff process* $\mathcal{U} = (\mathcal{U}_t)_{t \in [0, T]}$ are \mathbb{G} -adapted, càdlàg, $\mathbb{R} \cup \{+\infty\}$ -valued processes, such that $\mathcal{L}_T = \mathcal{U}_T$ and there exists a constant C such that

$$-C \leq \beta_t \mathcal{L}_t \leq \beta_t \mathcal{U}_t \quad \text{for } t \in [0, T]. \quad (2)$$

We refer to τ_c (respectively τ_p) as the moment of *call* (respectively *put*) of a game option.

Remarks 2.1 (i) In [15], the payoff processes \mathcal{L} and \mathcal{U} are implicitly assumed to be specified in relative terms with respect to a certain numéraire. In the present work, we prefer to make explicit the presence of the discount factor β .

(ii) Kallsen and Kühn [15] postulate that the lower payoff process \mathcal{L} is non-negative. However, as long as the process \mathcal{L} is bounded from below, all their results are applicable, by a simple shift argument.

(iii) One can deduce from (1) that we impose the priority of τ_p over τ_c , in the sense, that the effective

payment equals \mathcal{L}_{τ_p} (rather than \mathcal{U}_{τ_p}) on the event $\{\tau_p = \tau_c < T\}$. We thus follow here Kallsen and Kühn [15], from which we will borrow Theorem 2.1 below. Note, however, that in the general context of game options, this assumption is known to be essentially immaterial, in the sense, that it has no bearing neither on the price of a game option nor on the optimal stopping rules (cf. [17]).

Since we wish also to deal with the practically important case of a convertible bond with a positive call notice period, we need also to introduce a more general concept of an *extended game option*.

Definition 2.2 An *extended game option* is a contract with discounted payoff, as seen from the perspective of the holder, given by

$$\mathcal{J}(\tau_p, \tau_c) = \mathbf{1}_{\{\tau_p \leq \tau_c\}} \beta_{\tau_1} \mathcal{L}_{\tau_1} + \mathbf{1}_{\{\tau_p > \tau_c\}} \beta_{\tau_2} \mathcal{U}_{\tau_2}, \quad (3)$$

where $\tau_p, \tau_c \in \mathcal{G}_T^0$ are stopping times under the control of the holder and the issuer of an extended game option respectively, and $\tau_1, \tau_2 \in \{\tau_p, \tau_c\}$. Moreover, on the event $\{\tau_p \leq \tau_c\}$ we have $\tau_c \leq g(\tau_p)$ and on the event $\{\tau_c < \tau_p\}$ we have $\tau_p \leq h(\tau_c)$, where the Borel functions $g, h : [0, T] \rightarrow [0, T]$ are such that $g(t) \geq t, h(t) \geq t$ for $t \in [0, T]$. The *payoff processes* $\mathcal{L} = (\mathcal{L}_t)_{t \in [0, T]}$ and $\mathcal{U} = (\mathcal{U}_t)_{t \in [0, T]}$ are \mathbb{G} -adapted, càdlàg processes taking values in $\mathbb{R} \cup \{+\infty\}$.

In this rather abstract situation, we still find it convenient to refer to τ_c (respectively τ_p) as the moment of *call* (respectively *put*) of an extended game option. Thus if the contract is put by the holder prior to being called by the issuer (i.e., when $\tau_p \leq \tau_c$) then the holder receives \mathcal{L}_{τ_1} at time τ_1 ; otherwise, he receives \mathcal{U}_{τ_2} at time τ_2 . We may formally distinguish four classes of extended game options, corresponding to $\tau_1 = \tau_2 = \tau_p$, $\tau_1 = \tau_2 = \tau_c$, $\tau_1 = \tau_p$ and $\tau_2 = \tau_c$, or $\tau_1 = \tau_c$ and $\tau_2 = \tau_p$, respectively. The third class — for the choice of admissible functions g and h given as $g(t) = h(t) = T$ for $t \in [0, T]$ — corresponds to game options as defined above. Therefore, a game option can be seen as a special case of an extended game option.

2.2 Valuation of a Game Option

The concept of an arbitrage price of a game option can be introduced in various ways. Kallsen and Kühn [15] make the distinction between a *static* and a *dynamic* approach. The former point of view corresponds to the assumption that only a buy-and-hold strategy in the derivative asset is allowed, whereas the underlying asset can be traded dynamically. In the latter approach, it is assumed that a derivative asset becomes liquid and negotiable asset, so that it can be traded together with the underlying during the whole period $[0, T]$. Consequently, in a dynamic approach, in order to determine a price process of a derivative asset, it is postulated that the extended market, including this derivative asset, remains arbitrage-free. In this work, we shall adopt the dynamic point of view.

For the formal definition of a (dynamic) arbitrage price process of a game option, we refer the reader to Kallsen and Kühn [15, Definition 2.6]. As explained in their paper, this definition is based on an extension to markets containing game options of the *No Free Lunch with Vanishing Risk* condition, introduced by Delbaen and Schachermayer [12, Definition 2.8], using the notion of an admissible trading strategy involving assets in the underlying market and the game option. Without entering into details, let us only note that admissible strategies in this sense include, in particular, trading strategies in the underlying assets only, provided that the corresponding wealth process is bounded from below.

As a reality check of pertinency of Kallsen and Kühn's definition of an arbitrage price of a game option, we show in forthcoming papers [5, 6] that in more specific models, in which we are able to identify well determined processes as arbitrage prices in the sense of this definition, these processes can alternatively be characterized as a minimal super-hedging prices.

We decided not reproduce here the full statement of Definition 2.6 in [15], since it is rather technical and will not be explicitly used in the sequel. To proceed, it will be enough for us to

make use of the following characterization of an arbitrage price, in which \mathcal{M} stands for the class of all \mathbb{P} -equivalent martingale measures for the underlying market model driven by a family of \mathbb{G} -semimartingales.

Theorem 2.1 (Kallsen and Kühn [15, Theorem 2.9]) *Assume that*

$$\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \left(\sup_{t \in [0, T]} \beta_t \mathcal{L}_t \mid \mathcal{G}_0 \right) < \infty. \quad (4)$$

Then a process $J = (J_t)_{t \in [0, T]}$ is a discounted arbitrage price of a game option if and only if it satisfies the following two conditions:

- (i) *J is a \mathbb{G} -semimartingale,*
- (ii) *there exists $\mathbb{Q} \in \mathcal{M}$ such that J is the value process of the corresponding Dynkin game, in the sense that*

$$\begin{aligned} & \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\mathcal{J}(\tau_p, \tau_c) \mid \mathcal{G}_t) = J_t \\ & = \text{essinf}_{\tau_c \in \mathcal{G}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\mathcal{J}(\tau_p, \tau_c) \mid \mathcal{G}_t), \quad t \in [0, T]. \end{aligned} \quad (5)$$

An arbitrage price of a game option is thus equal to $\bar{J} = \beta^{-1}J$. This very general result essentially reduces the study of an arbitrage price of a game option to the study of the values under a martingale measure \mathbb{Q} of the corresponding family of Dynkin games with the issuer playing the role of the minimizer and the holder being the maximizer. Recall that the fact that the Dynkin game has a (conditional) *value* at time t means that we have equality between the lower value of the game, corresponding to the left-hand side of (5), and the upper value, as given by its right-hand side. It is well known that the lower value of a game is always less or equal to the upper value, but they do not need to coincide, in general.

In the proof of Theorem 2.9 in Kallsen and Kühn [15] (recalled as Theorem 2.1 above), condition (4) is a technical assumption that is used to prove the existence of an admissible self-financing strategy ζ in the traded assets (including the CS) with discounted wealth process greater than \mathcal{L} (see the inequality $v_0 + \int_0^t \theta_u dS_u \geq L_t$ for some real v_0 , in the notation of the proof of Theorem 2.9 in [15]). Note that in our multiplicative notation for the dividend adjustment on S , a self-financing portfolio built on cash and S means a portfolio with discounted wealth process βY such that

$$d(\beta_t Y_t) = \frac{\zeta_t}{Q_{t-}} d(\beta_t \hat{S}_t)$$

for some \mathbb{G} -predictable process ζ (see Section 1.3). Therefore, we have the following Corollary of Theorem 2.1.

Corollary 2.1 *Alternatively to condition (4), assume the existence of an initial wealth Y_0 and of a \mathbb{G} -predictable trading strategy ζ in S such that, for $t \in [0, T]$,*

$$\beta_t \mathcal{L}_t \leq \beta_0 Y_0 + \int_0^t \frac{\zeta_u}{Q_{u-}} d(\beta_u \hat{S}_u). \quad (6)$$

Then the characterization of discounted arbitrage prices of a game option in Theorem 2.1 is still valid.

A natural question arises whether it is possible to generalize Theorem 2.1 and/or Corollary 2.1 to the case of an extended game option. We shall not address this pertinent issue at this stage of our research, but we shall rather focus on specific subclasses of game options (convertible securities and, as a special case, convertible bonds with no call notice period) and extended game options (convertible bonds with a positive call notice period).

In this paper and the following ones [5, 6], the fact that in Corollary 2.1 we do not need to assume (4) is fundamental, as in the case of convertible bonds we will see in [6] that this condition fails to hold, even in the simplest model of defaultable stock.

3 Convertible Securities

Before stating a formal definition of a *convertible security* (CS), let us mention that by a convertible security we mean any financial contract that can be situated somewhere between a fairly general game option, as given by Definition 2.1, and a much more specific *convertible bond*, which will be discussed in some detail in Section 4.1 below.

3.1 Payoff Processes of a Convertible Security

In the next definition, we introduce a fairly general subclass of game options, called Convertible Securities (CS), which encompasses as special cases such financial instruments as convertible bonds and convertible preferred stocks. We wish to impose a restriction that a CS may not be called by its issuer on some random time interval $[0, \bar{\tau})$, where $\bar{\tau} \in \mathcal{G}_T^0$. This is implicitly enforced in Definition 3.1 by putting $\bar{U}_t = \infty$ on the random interval $[0, \bar{\tau})$.

Let an $\mathbb{R}_+ \cup \{\infty\}$ -valued \mathbb{G} -stopping time τ_d represent the *default time* of a reference entity.

Definition 3.1 A *convertible security* is a financial contract given as a game option with lower and upper payoff processes $\mathcal{L} = (\mathcal{L}_t)_{t \in [0, T]}$ and $\mathcal{U} = (\mathcal{U}_t)_{t \in [0, T]}$ representable in the following form:

$$\mathcal{L}_t = \chi_t + \mathbf{1}_{\{\tau_d > t\}} (\mathbf{1}_{\{t < T\}} L_t + \mathbf{1}_{\{t = T\}} \xi), \quad (7)$$

$$\mathcal{U}_t = \chi_t + \mathbf{1}_{\{\tau_d > t\}} (\mathbf{1}_{\{t < T\}} \bar{U}_t + \mathbf{1}_{\{t = T\}} \xi), \quad (8)$$

where:

- the *dividend process* χ is a real-valued \mathbb{G} -semimartingale with càdlàg sample paths and initial condition $\chi_0 = 0$,
- the *put payment* L is given as a \mathbb{G} -adapted, real-valued, càdlàg process on $[0, T]$,
- the process \bar{U} equals $\bar{U}_t = \mathbf{1}_{\{t < \bar{\tau}\}} \infty + \mathbf{1}_{\{t \geq \bar{\tau}\}} U_t$, where the lifting time of a *call protection* is modeled as a given stopping time $\bar{\tau} \in \mathcal{G}_T^0$, and where the *call payment* U is a \mathbb{G} -adapted, real-valued, càdlàg process on $[0, T]$, such that $L_t \leq \bar{U}_t$ for $t \in [0, \tau_d \wedge T)$, or equivalently,

$$L_t \leq U_t \quad \text{for } t \in [\tau_d \wedge \bar{\tau}, \tau_d \wedge T), \quad (9)$$

- the *payment at maturity* ξ is a \mathcal{G}_T -measurable real random variable, such that $L_T \leq \xi \leq U_T$.

In addition, the processes χ, L and the random variable ξ are assumed to satisfy the following inequalities, for some positive real numbers A, B, C :

$$\begin{aligned} -C &\leq \beta_t \chi_t \leq A + B \beta_{t \wedge \tau_d} \widehat{S}_{t \wedge \tau_d}, & t \in [0, T], \\ -C &\leq \beta_t L_t \leq A + B \beta_t \widehat{S}_t, & t \in [0, \tau_d \wedge T), \\ -C &\leq \beta_T \xi \leq A + B \beta_T \widehat{S}_T, \end{aligned} \quad (10)$$

where we recall that \widehat{S}_t represents the dividend-adjusted cash value of holding one stock at time t .

It follows from (10) that, for $t \in [0, T]$,

$$\beta_t \mathcal{L}_t \leq \left(A + B \beta_{t \wedge \tau_d} \widehat{S}_{t \wedge \tau_d} \right) + \mathbf{1}_{\{\tau_d > t\}} \left(\mathbf{1}_{\{t < T\}} (A + B \beta_t \widehat{S}_t) + \mathbf{1}_{\{t = T\}} (A + B \beta_T \widehat{S}_T) \right). \quad (11)$$

In the financial interpretation, the value χ_t represents the current value at time t of all ‘dividend payments’ on $[0, t]$ under the assumption that they are immediately reinvested in the savings account β . In a generic CS with recovery at default, there are no dividend payments after default, but this assumption is not needed, in general, and thus it is not made here. An explicit representation for the dividend process χ depends, of course, on the problem at hand. For instance, in Definition 4.1 we specify explicitly the dividend process of a convertible bond.

It is important to note that the component of payoff processes corresponding to the dividend process χ is the same in \mathcal{L} and \mathcal{U} , so that the associated component of the payoff \mathcal{J} only depends on τ_p and τ_c via their minimum $\tau = \tau_p \wedge \tau_c$, namely,

$$\mathcal{J}(\tau_p, \tau_c) = \beta_\tau \chi_\tau + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \left(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau = \tau_c < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau = T\}} \xi \right).$$

Also note that given the representation (7)–(8) for \mathcal{L} and \mathcal{U} , and for any bounded \mathbb{G} -semimartingale χ^b , we have the following alternative representation for \mathcal{L} and \mathcal{U}

$$\begin{aligned} \mathcal{L}_t &= (\chi_t - \chi_t^b + \mathbb{1}_{\{\tau_d \leq t\}} \chi_t^b) + \mathbb{1}_{\{\tau_d > t\}} \left(\mathbb{1}_{\{t < T\}} (L_t + \chi_t^b) + \mathbb{1}_{\{t = T\}} (\xi + \chi_T^b) \right), \\ \mathcal{U}_t &= (\chi_t - \chi_t^b + \mathbb{1}_{\{\tau_d \leq t\}} \chi_t^b) + \mathbb{1}_{\{\tau_d > t\}} \left(\mathbb{1}_{\{t < T\}} (\bar{U}_t + \chi_t^b) + \mathbb{1}_{\{t = T\}} (\xi + \chi_T^b) \right). \end{aligned}$$

This shows that the representation (7)–(8) is not unique, but is essentially defined up to a bounded \mathbb{G} -semimartingale χ^b .

3.2 Valuation of a Convertible Security

In the next results, we shall apply Corollary 2.1 in order to characterize both a *cum-dividend price* \bar{J} of a CS, as well as an *ex-dividend price* Π , where by definition $\Pi_t = \bar{J}_t - \chi_t$. Let us denote $\bar{\mathcal{G}}_T^t = \{\tau \in \mathcal{G}_T^t; \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}$.

Lemma 3.1 *A process $\bar{J} = (\bar{J}_t)_{t \in [0, T]}$ is a cum-dividend arbitrage price of a CS if and only if $\bar{J}_t = \beta_t^{-1} J_t$, where $J = (J_t)_{t \in [0, T]}$ is a \mathbb{G} -semimartingale and there exists $\mathbb{Q} \in \mathcal{M}$ such that J is the value process of the corresponding Dynkin game, in the sense that*

$$\begin{aligned} & \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}(\mathcal{J}(\tau_p, \tau_c) \mid \mathcal{G}_t) = J_t \\ & = \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\mathcal{J}(\tau_p, \tau_c) \mid \mathcal{G}_t), \quad t \in [0, T]. \end{aligned}$$

Proof. We shall apply Corollary 2.1 to the game option corresponding to the specification (7)–(8) of the payoff processes \mathcal{L} and \mathcal{U} a convertible security.

In the case of a convertible security, in order to show (6), it is enough, given inequality (11), to show the existence of an initial wealth Y_0 and of a \mathbb{G} -predictable trading strategy ζ in S such that

$$\begin{aligned} \beta_0 Y_0 + \int_0^t \frac{\zeta_u}{Q_{u-}} d(\beta_u \widehat{S}_u) &\geq (A + B\beta_{t \wedge \tau_d} \widehat{S}_{t \wedge \tau_d}) \\ &+ \mathbb{1}_{\{\tau_d > t\}} \left(\mathbb{1}_{\{t < T\}} (A + B\beta_t \widehat{S}_t) + \mathbb{1}_{\{t = T\}} (A + B\beta_T \widehat{S}_T) \right), \quad t \in [0, T]. \end{aligned} \tag{12}$$

But (12) is manifestly satisfied, provided that we take Y_0 large enough and we set $\zeta_t = \mathbb{1}_{\{t \leq \tau_d\}} B Q_{t-}$.

Hence, by an application of Corollary 2.1, we deduce that a process $J = (J_t)_{t \in [0, T]}$ is an arbitrage cum-dividend discounted price for a CS if and only if J is a \mathbb{G} -semimartingale, and there exists $\mathbb{Q} \in \mathcal{M}$ for which J is the value of the related Dynkin game, specifically

$$\begin{aligned} & \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\mathcal{J}(\tau_p, \tau_c) \mid \mathcal{G}_t) = J_t \\ & = \text{essinf}_{\tau_c \in \mathcal{G}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\mathcal{J}(\tau_p, \tau_c) \mid \mathcal{G}_t), \quad t \in [0, T], \end{aligned} \tag{13}$$

or equivalently, given the current specification (7)–(8) of \mathcal{L} and \mathcal{U} with $\bar{U}_t = \mathbb{1}_{\{t < \bar{\tau}\}} \infty + \mathbb{1}_{\{t \geq \bar{\tau}\}} U_t$,

$$\begin{aligned} & \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}(\mathcal{J}(\tau_p, \tau_c) \mid \mathcal{G}_t) = J_t \\ & = \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\mathcal{J}(\tau_p, \tau_c) \mid \mathcal{G}_t), \quad t \in [0, T]. \end{aligned} \tag{14}$$

This completes the proof of the lemma. \square

In the next result, we deal with an ex-dividend arbitrage price of a CS.

Theorem 3.1 *A process $\Pi = (\Pi_t)_{t \in [0, T]}$ is an ex-dividend arbitrage price of a CS if and only if Π is a \mathbb{G} -semimartingale and there exists $\mathbb{Q} \in \mathcal{M}$ such that Π satisfies*

$$\begin{aligned} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}(\pi_t(\tau_p, \tau_c) \mid \mathcal{G}_t) &= \Pi_t \\ &= \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\pi_t(\tau_p, \tau_c) \mid \mathcal{G}_t), \quad t \in [0, T], \end{aligned} \quad (15)$$

where the ex-dividend cash-flows $\pi_t(\tau_p, \tau_c)$ of a CS are defined by, for $t \in [0, T]$,

$$\beta_t \pi_t(\tau_p, \tau_c) = (\beta_\tau \chi_\tau - \beta_t \chi_t) + \mathbf{1}_{\{\tau_d > \tau\}} \beta_\tau \left(\mathbf{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbf{1}_{\{\tau = \tau_c < \tau_p\}} U_{\tau_c} + \mathbf{1}_{\{\tau = T\}} \xi \right)$$

with $\tau = \tau_p \wedge \tau_c$.

Proof. It is easy to check that we have, for any $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$,

$$\mathcal{J}(\tau_p, \tau_c) - \beta_t \chi_t = \beta_t \pi_t(\tau_p, \tau_c),$$

and thus

$$\beta_t^{-1} \mathbb{E}_{\mathbb{Q}}(\mathcal{J}(\tau_p, \tau_c) \mid \mathcal{G}_t) - \chi_t = \mathbb{E}_{\mathbb{Q}}(\pi_t(\tau_p, \tau_c) \mid \mathcal{G}_t).$$

Using Lemma 3.1, we conclude that a process Π is an ex-dividend arbitrage price of a CS if and only if (a) $\Pi_t + \chi_t$ is a \mathbb{G} -semimartingale, or equivalently Π is a \mathbb{G} -semimartingale, and (b) Π_t is the value of the Dynkin game with the cost criterion $\pi_t(\tau_p, \tau_c)$ over $\mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$. \square

Remarks 3.1 (i) It is clear that the random variable $\pi_t(\tau_p, \tau_c)$ is not \mathcal{G}_t -adapted for $t < T$. This is, of course, expected, since it represents the uncertain payments occurring between t and T .

(ii) We have that $\pi_t(\tau_p, \tau_c) = 0$ for any $t \geq \tau_d$. Therefore an ex-dividend arbitrage price of a CS is necessarily equal to 0, for $t \geq \tau_d$.

(iii) An ex-dividend price of a CS depends of course on the representation of the form (7)–(8) for \mathcal{L} and \mathcal{U} .

In what follows, by an arbitrage price we shall mean an ex-dividend arbitrage price, given a representation of the form (7)–(8) for \mathcal{L} and \mathcal{U} . A particular arbitrage price associated with some martingale measure \mathbb{Q} will be called the \mathbb{Q} -price of a CS.

Definition 3.2 A non-callable CS, referred to in the sequel as a PS (i.e., a *Puttable Security*), is a convertible security with $\bar{\tau} = T$, or, equivalently, $\bar{U} = \infty$. An *Elementary Security* (ES) is a PS admitting a representation (7)–(8) for \mathcal{L} and \mathcal{U} such that $\beta L = -C$ in (10).

We shall now apply Theorem 3.1 to a PS and an ES. Note that for a PS the value (15) becomes a \mathbb{Q} -Snell envelope with respect to stopping times $\tau_p \in \mathcal{G}_T^t$, whereas for an ES it reduces to a \mathbb{Q} -conditional expectation. This was, of course, expected since a PS is an American contingent claim, whereas an ES is easily seen to be a European contingent claim with terminal payoff ξ and dividend process χ .

Theorem 3.2 (i) *A process $\bar{\Pi} = (\bar{\Pi}_t)_{t \in [0, T]}$ is an arbitrage price of a PS if and only if $\bar{\Pi}$ is a \mathbb{G} -semimartingale and there exists $\mathbb{Q} \in \mathcal{M}$ such that*

$$\bar{\Pi}_t = \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\bar{\pi}_t(\tau_p) \mid \mathcal{G}_t), \quad t \in [0, T],$$

where the cash flows $\bar{\pi}_t(\tau_p)$ of a PS are defined by, for $t \in [0, T]$,

$$\beta_t \bar{\pi}_t(\tau_p) = (\beta_{\tau_p} \chi_{\tau_p} - \beta_t \chi_t) + \mathbf{1}_{\{\tau_d > \tau_p\}} \beta_{\tau_p} (\mathbf{1}_{\{\tau_p < T\}} L_{\tau_p} + \mathbf{1}_{\{\tau_p = T\}} \xi).$$

(ii) *A process $\Phi = (\Phi_t)_{t \in [0, T]}$ is an arbitrage price of an ES if and only if there exists $\mathbb{Q} \in \mathcal{M}$ such that*

$$\Phi_t = \mathbb{E}_{\mathbb{Q}}(\phi_t \mid \mathcal{G}_t), \quad t \in [0, T],$$

where the cash flows ϕ_t of an ES are defined by, for $t \in [0, T]$,

$$\beta_t \phi_t = (\beta_T \chi_T - \beta_t \chi_t) + \mathbf{1}_{\{\tau_d > T\}} \beta_T \xi.$$

Proof. Consistently with the notation, $\bar{\pi}_t(\tau_p)$ does not depend on τ_c and ϕ_t is independent of τ_p and τ_c . Part (i) follows by an application of Theorem 3.1. To prove (ii), note that, even though the process ϕ_t is not \mathbb{G} -adapted, in general, the process

$$\beta_0\phi_0 - \beta_t\phi_t = \beta_t\chi_t - \beta_0\chi_0, \quad t \in [0, T],$$

is a \mathbb{G} -semimartingale. Moreover, we have, for every $t \in [0, T]$,

$$\beta_t\Phi_t = \mathbb{E}_{\mathbb{Q}}(\beta_t\phi_t - \beta_0\phi_0 + \beta_0\phi_0 \mid \mathcal{G}_t) = \beta_t\phi_t - \beta_0\phi_0 + \mathbb{E}_{\mathbb{Q}}(\beta_0\phi_0 \mid \mathcal{G}_t),$$

so that Φ is a \mathbb{G} -semimartingale as well. Once again, we conclude by applying Theorem 3.1. \square

4 Convertible Bonds

We shall now address the issue of arbitrage valuation of a convertible bond with real-life features, including the call notice period. As will be explained in Section 4.4, it is rather difficult to directly value a convertible bond with a positive call notice period, since it does not fulfil conditions of the definition of a game option, in general. To circumvent this difficulty, we shall introduce the concept of a *reduced convertible bond* (RB), that is, a convertible bond whose value upon call is exogenously given as some stochastic process, so that we can assume, without loss of generality, that the bond has no call notice period. Since an RB is a special case of a CS, the valuation results of Section 3.2 can be directly applied to this class of convertible bonds.

4.1 Covenants of a Convertible Bond

To describe the covenants of a typical *convertible bond* (CB), we introduce the following additional notation:

\bar{N} : the par (nominal) value,

$T_i, c_i, i = 0, 1, \dots, K$: the coupon dates T_0, \dots, T_K are deterministic fixed times with $T_0 = 0$ (for notational convenience) and $T_{K-1} < T \leq T_K$; the coupon amounts c_i are $\mathcal{G}_{T_{i-1}}$ -measurable and bounded, for $i = 1, 2, \dots, K$, and $c_0 = 0$,

A_t : the accrued interest at time t , specifically,

$$A_t = \frac{t - T_{i_t-1}}{T_{i_t} - T_{i_t-1}} c_{i_t},$$

where i_t is the integer satisfying $T_{i_t-1} \leq t < T_{i_t}$; in view of our assumptions on the coupons, the process $(A_t)_{t \in [0, T]}$ is càdlàg and \mathbb{G} -adapted,

\bar{R}_t : the recovery rate upon default of the issuer at time t , a \mathbb{G} -adapted bounded process,

\bar{X}_t : the claim upon default of the issuer at time t , a \mathbb{G} -adapted bounded process,

κ : the conversion factor,

$R_t^{cb} = \bar{R}_t \bar{X}_t \vee \kappa S_t$: the recovery process,

χ_t^{cb} : the dividend process (to be specified below),

$\xi^{cb} = \bar{N} \vee \kappa S_T + A_T$: the payoff at maturity,

$\bar{P} \leq \bar{C}$: the put and call nominal payments, respectively,

$\delta \geq 0$: the length of the call notice period (see the detailed description below),

$t^\delta = (t + \delta) \wedge T$: the end date of the call notice period started at t .

We shall now present a formal description of specific CB covenants. Let us consider a CB at any date $t \in [0, T]$ at which it is still alive. Then we have the following provisions:

put/conversion provision – at any time $\tau_p \in [t, \tau_c \wedge \tau_d \wedge T]$, where τ_c is a stopping time under the discretion of the issuer, the bond holder may convert a CB to κ shares of equity. In addition, at any time $\tau_p \in [t, \tau_c \wedge \tau_d \wedge T]$, and possibly also at τ_c if $\tau_c < \tau_d \wedge T$, the holder may put (return) the bond to the issuer for a nominal put payment \bar{P} pre-agreed at time of issuance. Only one of the two above decisions may be executed. Since the bond holder is also entitled to receive a relevant accrued interest payment, the *effective put/conversion payment* collected in case of put or conversion (depending on which one is more favorable to the holder) at time τ_p (if $\tau_p < T$) equals $L_{\tau_p}^{cb} = \bar{P} \vee \kappa S_{\tau_p} + A_{\tau_p}$, where κ denotes the conversion ratio. The effective put payment in case $\tau_p = T$ is considered separately (see the *promised payment* below).

call provision – the issuer has the right to call the bond at any time $\tau_c \in [t, \tau_p \wedge \tau_d \wedge T]$, where τ_p is a random time under the discretion of the holder, for a nominal call payment \bar{C} pre-agreed at time of issuance. More precisely, there is a fixed call notice period $\delta \geq 0$ (typically, one month) such that if the issuer calls the bond at time τ_c , then the bond holder has either to redeem the bond for \bar{C} or convert the bond into κ shares of stock, at any time u at its convenience in $[\tau_c, \tau_c^\delta]$, where $\tau_c^\delta = (\tau_c + \delta) \wedge T$. Accounting for accrued interest, the *effective call/conversion payment* to the holder at time u is $\bar{C} \vee \kappa S_u + A_u$.

call protection – typically, a CB also includes *call protections*, either *hard* or *soft*. For instance, the issuer's right to call a CB early becomes active only after a certain period of time has lapsed since the original issue date. A CB, which can't be called under any circumstances during the initial time period $[0, \bar{T})$, is subject to *hard* call protection. Alternatively, a CB that is non-callable unless the stock price reaches a certain predetermined level, say \bar{S} , is subject to *soft* call protection. The introduction of the stopping times $\bar{\tau}$ in \mathcal{G}_T^0 , and of the associated class $\bar{\mathcal{G}}_T^t \subseteq \mathcal{G}_T^t$, allows one to model quite general kinds of call protections. So hard call protections correspond to $\tau_c \in \bar{\mathcal{G}}_T^t$ with $\bar{\tau} = \bar{T}$, and standard soft call protections to $\tau_c \in \bar{\mathcal{G}}_T^t$ with $\bar{\tau} = \inf\{t \in \mathbb{R}_+; S_t \geq \bar{S}\} \wedge T$.

promised payment – the issuer agrees to pay to the bond holder, at any coupon date T_i prior to default time τ_d and to $\tau_p \wedge \tau_c \wedge T$, a bounded coupon amount c_i . He also agrees to pay the par value \bar{N} at the maturity date T , provided that $T < \tau_d$ and $T \leq \tau_p \wedge \tau_c$. Since the bond holder may still convert at time T , we define the *effective payment at maturity* as $\xi^{cb} = \bar{N} \vee \kappa S_T + A_T$; it is collected at time T if the CB is still alive at T .

recovery structure at default – it is assumed throughout that in the case of default at time $\tau_d \leq \tau_p \wedge \tau_c \wedge T$, the *effective recovery* $R_{\tau_d}^{cb} = \bar{R}_{\tau_d} \bar{X}_{\tau_d} \vee \kappa S_{\tau_d}$ is recovered. Indeed, we assume that the CB can still be converted at default time τ_d .

It is typically assumed that $\bar{P} \leq \bar{N} \leq \bar{C}$, which we suppose in the following.

Remarks 4.1 (i) As specified above, at maturity the bond holder is allowed to convert, but not to put, the bond. Some authors allow for a put decision at maturity date as well. In fact, allowing put decisions at maturity would not change anything, as long as one supposes (as we do) that $\bar{P} \leq \bar{N}$. Indeed, if $\bar{P} \leq \bar{N}$, we have $N_T = (\bar{N} \vee \kappa S_T) + A_T = (\bar{P} \vee \bar{N} \vee \kappa S_T) + A_T$.

(ii) An important issue in the valuation of a CB is the so-called *dilution effect*. Dilution is the fact that the equity price may drop upon conversion, due to the sudden increase of the number of shares in circulation [10]. In practice, the importance of this effect depends on the number of bond holders, who decide to convert simultaneously. In our framework, we deal with a representative holder, who is supposed to make optimal decisions. Therefore, the whole issue of the convertible bond will be converted at the same time, so that a jump in the stock price upon conversion is expected. To account for dilution, one could introduce a fractional loss $0 \leq \nu \leq 1$ of the stock price at put, so that $\bar{P} \vee \kappa S_{\tau_p} = \bar{P} \vee \kappa(1 - \nu)S_{\tau_p}$. However, in the abstract framework considered in this paper, this would be immaterial.

(iii) A further possible covenant of a CB is *resettability*. *Resettability* means that to compensate for fluctuations in S , the conversion ratio κ may depend on S_t in a particular way specified in the bond indenture. It is straightforward to check that all the results in this paper remain valid, if one assumes that $\kappa_t = \kappa(S_t)$ for some bounded Borel function κ .

(iv) There exist soft call protection clauses more sophisticated than the one mentioned above, such as clauses preventing the issuer to call a CB unless the stock has been above a certain level for a given amount of time. A soft call protection always introduces a certain path-dependency to the valuation problem (cf. [2, 19]). However, we shall see that it does not complicate much the analysis from a general point of view. Naturally, it makes computationally heavier the numerical resolution of the pricing variational inequalities in a Markovian model (see [6]).

(v) In practice, coupons of a CB are paid discretely, rather than continuously. Frequently, in the literature on CBs, an assumption is made that coupons are paid continuously. Assumption of continuously paid coupons is not necessary however; it is imposed in order to slightly simplify the analysis of cash flows and price processes. It could of course be dealt with in our set-up if wished, as it can be seen on the general definition of the dividend process of a CS (see Definition 3.1).

(vi) A default claim process \bar{X} is specified in the indenture of a CB. Typically, \bar{X} is simply equal to the bond par value, or the bond par value plus the accrued interest. A recovery rate \bar{R} depends on legal specifications, such as the seniority of the related debt, etc. In practice, \bar{R} tends to be lower in periods with more defaults. However, this statistical observation holds under the real-world probability, with no obvious consequences under the market pricing measure [4]. A common recovery assumption is the so-called *face recovery* assumption, which means that \bar{R} is a given constant (typically, $\bar{R} = 40\%$ for investment grade issues) and that \bar{X} is equal to \bar{N} .

(vii) Upon default, the stock price process typically falls sharply. To account for this effect, one should introduce, in a model for the stock process S , a fractional loss upon default $0 \leq \eta \leq 1$, such that $S_{\tau_d} = (1 - \eta)S_{\tau_d-}$ (see, for instance, [5]). However, in the abstract framework considered in this paper, this particular feature of the stock price is irrelevant.

Definition 4.1 In accordance with the CB covenants, the dividend process χ^{cb} of a CB is given by the expression

$$\beta_t \chi_t^{cb} = \sum_{0 \leq T_i \leq t, T_i < \tau_d} \beta_{T_i} c_i + \mathbb{1}_{\{0 \leq \tau_d \leq t\}} \beta_{\tau_d} R_{\tau_d}^{cb}, \quad t \in [0, T]. \quad (16)$$

Recall that we write

$$R_{\tau_d}^{cb} = \bar{R}_{\tau_d} \bar{X}_{\tau_d} \vee \kappa S_{\tau_d}, \quad L_t^{cb} = \bar{P} \vee \kappa S_t + A_t, \quad \xi^{cb} = \bar{N} \vee \kappa S_T + A_T.$$

Thus the process χ defined by (16) is a \mathbb{G} -semimartingale, as a piecewise constant \mathbb{G} -adapted process (up to the factor β). Moreover, since the discount factor β and the dividend adjustment factor Q are positive and locally bounded processes with Q being locally bounded away from zero, the technical conditions (10) are satisfied by χ^{cb} , L^{cb} and ξ^{cb} .

4.2 Convertible Bonds without Call Notice Period

Assume first that a CB has no call notice period, so that $\delta = 0$. In this special case, a CB is a particular example of a CS.

Definition 4.2 A *convertible bond with no call notice period* is a convertible security with the lower and upper payoff processes \mathcal{L}^{cb} and \mathcal{U}^{cb} given by the expressions

$$\mathcal{L}_t^{cb} = \chi_t^{cb} + \mathbb{1}_{\{\tau_d > t\}} (\mathbb{1}_{\{t < T\}} L_t^{cb} + \mathbb{1}_{\{t = T\}} \xi^{cb}), \quad (17)$$

$$\mathcal{U}_t^{cb} = \chi_t^{cb} + \mathbb{1}_{\{\tau_d > t\}} (\mathbb{1}_{\{t < T\}} \bar{U}_t^{cb} + \mathbb{1}_{\{t = T\}} \xi^{cb}), \quad (18)$$

where we set

$$\bar{U}_t^{cb} = \mathbb{1}_{\{t < \bar{\tau}\}} \infty + \mathbb{1}_{\{t \geq \bar{\tau}\}} (\bar{C} \vee \kappa S_t + A_t), \quad t \in [0, T]. \quad (19)$$

Remarks 4.2 Note that in view of our formulation of the problem the put/conversion or call decisions may take place after the default time τ_d . Nevertheless, the discounted payoff processes

$\beta\mathcal{L}^{cb}$ and $\beta\mathcal{U}^{cb}$ are constant on the set $\{t \geq \tau_d\}$ (note that the process $\beta\chi^{cb}$ is stopped at τ_d). Thus, effectively, the convertible bond game is stopped at the default time τ_d , unless the decision to stop the game was made earlier than that.

The arbitrage valuation of a CB with no call notice period is covered by Theorem 3.1. Therefore, it remains to address the issue of valuation of convertible bonds with a positive call notice period.

4.3 Convertible Bonds with a Positive Call Notice Period

At the intuitive level, a convertible bond with a positive call notice period $\delta > 0$ can be seen as a contract involving the following decisions: the decision to call the bond by its issuer, the decision to put/convert a bond by its holder, provided that the bond was not yet called, and the decision to put/convert the bond by its holder during the call notice period $[t, t + \delta]$, assuming that the bond has been called at some date t . This rather loose description is made rigorous in the following definition.

Definition 4.3 A *convertible bond with a positive call notice period* is an extended game option with discounted payoff, as seen from the perspective of the holder, given by

$$\mathcal{J}(\tau_p, \tau_c) = \mathbb{1}_{\{\tau_p \leq \tau_c\}} \beta_{\tau_p} \mathcal{L}_{\tau_p}^{cb} + \mathbb{1}_{\{\tau_p > \tau_c\}} \beta_{\tau_p} \mathcal{L}_{\tau_p}^c, \quad (20)$$

where τ_c, τ_p are \mathbb{G} -stopping time under the control of the issuer and the holder respectively, and $\tau_p \leq (\tau_c + \delta) \wedge T$ on the event $\{\tau_c < \tau_p\}$. Moreover, \mathcal{L}^{cb} is given by (17) and \mathcal{L}^c is given by the formula

$$\mathcal{L}_t^c = \chi_t^{cb} + \mathbb{1}_{\{\tau_d > t\}} (\bar{C} \vee \kappa S_t + A_t). \quad (21)$$

On the one hand, it is worth noting that for $\delta = 0$ the definition above formally excludes $\tau_p > \tau_c$, so that formula (20) reduces to $\mathcal{J}(\tau_p, \tau_c) = \beta_{\tau_p} \mathcal{L}_{\tau_p}^{cb}$. Consequently, for $\delta = 0$, Definition 4.3 (which would be inappropriate in this case) does not reduce to Definition 4.2.

On the other hand, in order to obtain the definition above, it suffices to set $\tau_1 = \tau_2 = \tau_p$, $g(t) = T$, $h(t) = (t + \delta) \wedge T$, $\mathcal{L} = \mathcal{L}^{cb}$ and $\mathcal{U} = \mathcal{L}^c$ in Definition 2.2 of an extended game option.

As already mentioned, we do not attempt here to directly value a convertible bond with a positive call notice, in the sense that we do not adapt Theorem 2.1 to the case of extended game options.

4.4 Reduced Convertible Bonds

An alternative approach proposed in this work is based on a conjecture that the valuation of a convertible bond with positive call notice period can be done recursively. In the first step, we shall value this bond upon call. In the second step, we use this price as the payoff at call time of a CB with no call notice period. This idea motivates the introduction of the following auxiliary concept.

Definition 4.4 A *reduced convertible bond* (RB) is a convertible security with the lower payoff process \mathcal{L}^{cb} given by (17) and the upper payoff process \mathcal{U}^{cb} given by (18) with

$$\bar{U}_t^{cb} = \mathbb{1}_{\{t < \bar{\tau}\}} \infty + \mathbb{1}_{\{t \geq \bar{\tau}\}} U_t^{cb}, \quad t \in [0, T],$$

where $(U_t^{cb})_{t \in [0, T]}$ is a càdlàg process that is required to satisfy the following inequality

$$U_t^{cb} \geq \bar{C} \vee \kappa S_t + A_t, \quad t \in [0, T]. \quad (22)$$

The financial interpretation of the process U^{cb} is that U_t^{cb} represents the value of our reduced convertible bond upon a call at time t . As we shall see in what follows, one of the major problems

arising here is that an arbitrage price of a CB upon call is not uniquely defined, unless we place ourselves in the framework of a complete market model.

Note that the only difference between Definitions 4.2 and 4.4 is that the latter definition postulates that the process U^{cb} is exogenous and satisfies inequality (22), whereas in the former this process is given by equality (19). It is thus trivial to observe that a CB with no call notice period is an RB. The same remark applies to a *puttable bond* (PB), that is, a CB with no call clause (formally, we set $\bar{\tau} = T$).

Since an RB is a CS, in order to obtain a characterization of an arbitrage price of an RB, it suffices to make use of Theorem 3.1, with the properly modified notation.

In the next section, we shall examine a method of interpreting and valuing a CB with positive call notice period as an RB, based on an endogenous specification of the random variable U_t^{cb} as arbitrage price of a certain PB starting at time t .

4.5 Valuation of a Convertible Bond upon Call

It follows from (20) and (21) (see also Kwok and Lau [19]) that when a convertible bond with a positive call notice period is called at some date t , it is actually replaced by a puttable bond. This particular PB, which will be referred to as the t -PB in what follows, is endowed with the same characteristics as a considered CB, except that:

- (i) the inception date of the t -PB is t , its maturity is $t^\delta = (t + \delta) \wedge T$, and its nominal is equal to the call payment \bar{C} ,
- (ii) the coupon schedule of the t -PB is the trace on $(t, t^\delta]$ of the coupon schedule of a CB,
- (iii) the effective put/conversion payment of the t -PB is equal to the effective call/conversion payment $\bar{C} \vee \kappa S_u + A_u$ of a CB, at any date $u \in [t, t^\delta]$.

In (ii), we excluded t from the coupon schedule of the t -PB, because any coupon falling at call time is already paid to the bond holder via the CB.

Definition 4.5 A t -PB is a puttable bond with the inception date t , the maturity date t^δ , and the cash flows $\bar{\pi}_u^t(\tau_p)$, $u \in [t, t^\delta]$, given by

$$\beta_u \bar{\pi}_u^t(\tau_p) = (\beta_{\tau_p} \chi_{\tau_p}^{cb} - \beta_u \chi_u^{cb}) + \mathbf{1}_{\{\tau_d > \tau_p\}} \beta_{\tau_p} (\bar{C} \vee \kappa S_{\tau_p} + A_{\tau_p}),$$

where τ_p belongs to $\mathcal{G}_{t^\delta}^u$, that is, τ_p is a \mathbb{G} -stopping time taking values in $[u, t^\delta]$.

As we already noted, any PB can be seen as an example of an RB, and thus as an example of a CS. Hence, we may apply Theorem 3.2 in order to establish the following auxiliary result.

Proposition 4.1 Let us fix $t \in [0, T]$. Assume that $(\bar{\Pi}_u^t)_{u \in [t, t^\delta]}$ is an arbitrage-free price of the t -PB. Then $\bar{\Pi}_t^t \geq \bar{C} \vee \kappa S_t + A_t$ on the event $\{\tau_d > t\}$.

Proof. By part (i) in Theorem 3.2, there exists $\mathbb{Q} \in \mathcal{M}$ such that

$$\bar{\Pi}_t^t = \text{esssup}_{\tau_p \in \mathcal{G}_{t^\delta}^t} \mathbb{E}_{\mathbb{Q}}(\bar{\pi}_t^t(\tau_p) \mid \mathcal{G}_t). \quad (23)$$

By considering the specific stopping time $\tau_p = t$ in the left-hand side of (23), we obtain the inequality $\bar{\Pi}_t^t \geq \bar{C} \vee \kappa S_t + A_t$ on the event $\{\tau_d > t\}$. \square

Assume that for any $t \in [0, T]$, the price process $\bar{\Pi}^t$ is arbitrage-free. Since the price processes $(\bar{\Pi}_u^t)_{u \in [t, t^\delta]}$ of t -PBs constitute a whole family of processes indexed by $t \in [0, T]$, this assumption means that each of these price processes is arbitrage-free, in the sense of Definition 2.6 in [15].

It is not clear, however, whether the family of random variables $\bar{\Pi}_t^t$, $t \in [0, T]$, can be considered as a well-defined càdlàg process. Let us make a bold assumption that this is indeed the case. Then inequality (22) is manifestly satisfied by the process

$$U_t^{cb} = \mathbf{1}_{\{\tau_d > t\}} \bar{\Pi}_t^t + \mathbf{1}_{\{\tau_d \leq t\}} (\bar{C} \vee \kappa S_t + A_t).$$

In this way, we implement our conjecture of reducing the valuation problem for a CB with a positive call notice period to the already solved case of valuation of an RB.

5 Decomposition of Convertible Securities

We shall now study the pertinent decompositions of payoffs and prices of convertible securities. In particular, we will provide a decomposition of a reduced convertible bond into a *bond component* and a *game option component*, which will allow us to give a definite meaning to commonly used terms of the ‘CB spread’ and the ‘CB implied volatility’ (see, for instance, Connolly [10]). To motivate this issue, let us first examine relevant market data.

5.1 Spread and Implied Volatility of a Convertible Bond

To motivate this point, let us consider some relevant market data (data provided by courtesy of Credit Agricole, Paris).

CB	Stock Price	Nominal	CB Price	Credit Spread	CB IV
Alcatel 4.75% Jan-11	8.39	16.18	17.42	135 bp	30.2%
Pinault 2.50% Jan-08	77.80	90.97	93.98	65 bp	21.5%
Cap Gemini 2.00 % Jun-09	25.25	39.86	41.80	65 bp	33.9%

Table 1: CB data on names of the CAC40 on May 10, 2005

Table 1 provides market quotes on CBs issued by the three companies of the CAC40 (French stock index) on May 10, 2005. The CB prices are Mid-Market Trading Euro Prices and *CB implied volatilities* (CB IV) are Offer-Side Implied Volatilities. In accordance with the French convention for quoting CBs, the bonds’ nominals in Table 1 have been scaled by a factor κ^{-1} , so that the data in Table 1 correspond to a conversion ratio κ equal to 1. For instance, the price of the scaled Alcatel CB is equal to 17.42 euros. Immediate conversion would be for one share of stock priced at 8.39 euros, and the scaled nominal of the CB is equal to 16.18 euros.

CB	CB Expected Life	Option Strike and Expiry	Option IV
Alcatel 4.75% Jan-11	Oct-10	13.0 Dec-09	30.7%
Pinault 2.50% Jan-08	Nov-07	90.0 Dec-07	20.5%
Cap Gemini 2.00 % Jun-09	May-09	40.0 Dec-08	35.6%

Table 2: CBs and the closest listed options

For comparison, Table 2 shows market quotes on the closest listed option for each case considered in Table 1. The ‘closest listed option’ means the listed vanilla option with strike and maturity as close as possible to the scaled nominal and to the ‘CB expected life’, i.e. the most likely time of call, put, conversion or default, as forecasted by financial analysts.

Investors are expected to use the information in Tables 1 and 2 to assess relative value of CBs and options, and to take positions as a consequence. For instance, in some circumstances traders used to say that buying a CB is a ‘cheap way to buy volatility’. This means that in their view,

the option component of a CB is ‘cheaper’ (has a lower Black-Scholes implied volatility) than the corresponding listed vanilla option. It is thus a bit surprising that, to the best of our knowledge, the exact meaning of a ‘CB spread’ and a ‘CB implied volatility’ (CB IV in Table 1) has not been yet specified in the literature.

The decomposition of a CB into a bond and option components is well known in the ‘exchange option’ case when the conversion can only occur at maturity and there are no put or call clauses [21], but not in the general case of a defaultable CB with call and/or put covenants. In particular, at the intuitive level, it seems clear that the strike of the option embedded into a general convertible bond is a floating strike, equal to the price of a defaultable bond. So, how the implied volatility for the CB (if well-defined) in Table 1 and the corresponding implied volatility for the closest listed option in Table 2 should be compared is definitely far from being obvious.

5.2 Decomposition of Cash Flows of a Convertible Security

We first consider a convertible security corresponding to the data set $(\chi, L, U, \bar{\tau}, \xi)$, as defined in Section 3.1. Let also be given a bounded elementary security, namely an ES with the cash flows given by the expression

$$\beta_t \phi_t = (\beta_T \chi_T^b - \beta_t \chi_t^b) + \mathbf{1}_{\{\tau_d > T\}} \beta_T \xi^b, \quad (24)$$

where the dividend process χ^b and the payment at maturity ξ^b are assumed to be bounded. Given a probability measure $\mathbb{Q} \in \mathcal{M}$, we define the \mathbb{Q} -price of this ES as $\Phi_t = \mathbb{E}_{\mathbb{Q}}(\phi_t | \mathcal{G}_t)$, for $t \in [0, T]$. Note that by Theorem 3.2(ii), the process Φ is actually an arbitrage price for the ES, associated with the martingale measure \mathbb{Q} .

Lemma 5.1 (i) *The cash flows of the CS can be decomposed as follows:*

$$\pi_t(\tau_p, \tau_c) = \phi_t + \varphi_t(\tau_p, \tau_c), \quad t \in [0, T], \quad (25)$$

where ϕ_t represents the cash flows of the bounded ES, and $\varphi_t(\tau_p, \tau_c)$ is given by the formula

$$\begin{aligned} \beta_t \varphi_t(\tau_p, \tau_c) &= (\beta_\tau (\chi_\tau - \chi_\tau^b) - \beta_t (\chi_t - \chi_t^b)) \\ &+ \mathbf{1}_{\{\tau_d > \tau\}} \beta_\tau \left(\mathbf{1}_{\{\tau = \tau_p < T\}} (L_{\tau_p} - \phi_{\tau_p}) + \mathbf{1}_{\{\tau = \tau_c < \tau_p\}} (U_{\tau_c} - \phi_{\tau_c}) + \mathbf{1}_{\{\tau = T\}} (\xi - \xi^b) \right). \end{aligned} \quad (26)$$

(ii) *Let a martingale measure $\mathbb{Q} \in \mathcal{M}$ be given. Then we have*

$$\mathbb{E}_{\mathbb{Q}}(\varphi_t(\tau_p, \tau_c) | \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(\psi_t(\tau_p, \tau_c) | \mathcal{G}_t), \quad t \in [0, T], \quad (27)$$

where $\psi_t(\tau_p, \tau_c)$ is defined by

$$\begin{aligned} \beta_t \psi_t(\tau_p, \tau_c) &= (\beta_\tau (\chi_\tau - \chi_\tau^b) - \beta_t (\chi_t - \chi_t^b)) \\ &+ \mathbf{1}_{\{\tau_d > \tau\}} \beta_\tau \left(\mathbf{1}_{\{\tau = \tau_p < T\}} (L_{\tau_p} - \Phi_{\tau_p}) + \mathbf{1}_{\{\tau = \tau_c < \tau_p\}} (U_{\tau_c} - \Phi_{\tau_c}) + \mathbf{1}_{\{\tau = T\}} (\xi - \xi^b) \right). \end{aligned} \quad (28)$$

Proof. The decomposition of cash flows stated in part (i) is straightforward. For part (ii), we recall from the proof of part (ii) in Theorem 3.2 that

$$\beta_{\tau_p} \Phi_{\tau_p} = \beta_{\tau_p} \phi_{\tau_p} - \beta_0 \phi_0 + \mathbb{E}_{\mathbb{Q}}(\beta_0 \phi_0 | \mathcal{G}_{\tau_p}) = \mathbb{E}_{\mathbb{Q}}(\beta_{\tau_p} \phi_{\tau_p} | \mathcal{G}_{\tau_p}),$$

where the second equality holds since the random variable $\beta_{\tau_p} \phi_{\tau_p} - \beta_0 \phi_0$ is \mathcal{G}_{τ_p} -measurable. Hence, by taking iterated conditional expectations, we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{\{\tau < \tau_d\}} \mathbf{1}_{\{\tau = \tau_p < T\}} \beta_\tau \phi_{\tau_p} | \mathcal{G}_t) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{\{\tau_p < \tau_d\}} \mathbf{1}_{\{\tau = \tau_p < T\}} \beta_{\tau_p} \phi_{\tau_p} | \mathcal{G}_{\tau_p}) \middle| \mathcal{G}_t\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\mathbf{1}_{\{\tau_p < \tau_d\}} \mathbf{1}_{\{\tau = \tau_p < T\}} \beta_{\tau_p} \mathbb{E}_{\mathbb{Q}}(\phi_{\tau_p} | \mathcal{G}_{\tau_p}) \middle| \mathcal{G}_t\right) \\ &= \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{\{\tau < \tau_d\}} \mathbf{1}_{\{\tau = \tau_p < T\}} \beta_\tau \Phi_{\tau_p} | \mathcal{G}_t) \end{aligned}$$

where we have used the fact that the random variable $\mathbb{1}_{\{\tau_p < \tau_d\}} \mathbb{1}_{\{\tau = \tau_p < T\}} \beta_{\tau_p}$ is \mathcal{G}_{τ_p} -measurable. Applying similar arguments, we also get

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau < \tau_d\}} \mathbb{1}_{\{\tau = \tau_c < \tau_p\}} \beta_{\tau} \phi_{\tau_c} \mid \mathcal{G}_t) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\tau_c < \tau_d\}} \mathbb{1}_{\{\tau = \tau_c < \tau_p\}} \beta_{\tau_c} \mathbb{E}_{\mathbb{Q}}(\phi_{\tau_c} \mid \mathcal{G}_{\tau_c}) \mid \mathcal{G}_t\right) \\ &= \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau < \tau_d\}} \mathbb{1}_{\{\tau = \tau_c < \tau_p\}} \beta_{\tau} \Phi_{\tau_c} \mid \mathcal{G}_t). \end{aligned}$$

It is now easily seen that equality (27) is valid. \square

We define the \mathbb{Q} -exchange CS, as the CS with dividend process $\chi - \chi^b$, put payment $L_t - \Phi_t$, call payment $U_t - \Phi_t$, call protection lifting time $\bar{\tau}$, and payment at maturity $\xi - \xi^b$. Consequently, the \mathbb{Q} -exchange CS is a CS with cash flows $\psi_t(\tau_p, \tau_c)$ given by (28).

Remarks 5.1 (i) Note that the cash flows of the \mathbb{Q} -exchange CS depend on the choice of \mathbb{Q} , through the definition of the price process Φ of the ES.

(ii) Since the process Φ is an arbitrage price for the ES, the \mathbb{Q} -exchange CS can be interpreted as a game option to exchange the ES for either L or U (as seen from the perspective of the holder), according to which player decides first to stop the game.

5.3 Price Decomposition of a Convertible Security

We are now ready to decompose the price of a CS. Assume that we are given a CS and a bounded ES as in the previous section.

Theorem 5.1 *Let $\mathbb{Q} \in \mathcal{M}$ be a martingale measure, and let Φ be the arbitrage \mathbb{Q} -price of the ES.*

(i) *If Π is an arbitrage \mathbb{Q} -price for the CS, then $\Psi = \Pi - \Phi$ is an arbitrage \mathbb{Q} -price for the \mathbb{Q} -exchange CS.*

(ii) *If Ψ is an arbitrage \mathbb{Q} -price for the \mathbb{Q} -exchange CS then $\Pi = \Phi + \Psi$ is an arbitrage \mathbb{Q} -price for the CS.*

Proof. To prove (i), note that, using (15) and (25), we obtain

$$\begin{aligned} \Psi_t = \Pi_t - \Phi_t &= \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}(\pi_t(\tau_p, \tau_c) \mid \mathcal{G}_t) - \mathbb{E}_{\mathbb{Q}}(\phi_t \mid \mathcal{G}_t) = \\ &= \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}(\varphi_t(\tau_p, \tau_c) \mid \mathcal{G}_t) = \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}(\psi_t(\tau_p, \tau_c) \mid \mathcal{G}_t) \end{aligned}$$

where the last equality is a consequence of (27). Moreover, the difference $\Psi = \Pi - \Phi$ is obviously a \mathbb{G} -semimartingale. Thus Ψ is an arbitrage \mathbb{Q} -price for the \mathbb{Q} -exchange CS, by the ‘if’ part of Theorem 3.1. The proof of part (ii) is similar to that of part (i). \square

Corollary 5.1 *Under the assumptions of Theorem 5.1(i) or 5.1(ii), we have*

(i) *If the process $\beta_t \chi_t^b$ is non-decreasing and $\xi^b \geq 0$, then $\Phi \geq 0$, and thus $\Pi \geq \Psi$,*

(ii) *If the process $\beta_t(\chi_t - \chi_t^b)$ is non-decreasing, $U \geq \Phi$ on $[\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$, and $\xi \geq \xi^b$, then $\Psi \geq 0$, and thus $\Pi \geq \Phi$.*

Proof. (i) If the process $\beta_t \chi_t^b$ is non-decreasing and $\xi^b \geq 0$, then $\phi \geq 0$, and thus obviously $\Phi \geq 0$. Consequently, we have that $\Pi = \Phi + \Psi \geq \Psi$.

(ii) Let us show that at any time $t < \tau_d \wedge T$ such that $L_t < \Phi_t$, exchanging the ES for the payoff L_t is suboptimal for the holder of the \mathbb{Q} -exchange CS. Towards this end, we define $\check{\psi}_t(\tau_p, \tau_c)$ by the formula

$$\begin{aligned} \beta_t \check{\psi}_t(\tau_p, \tau_c) &= (\beta_{\tau}(\chi_{\tau} - \chi_{\tau}^b) - \beta_t(\chi_t - \chi_t^b)) \\ &\quad + \mathbb{1}_{\{\tau_d > \tau\}} \beta_{\tau} \left(\mathbb{1}_{\{\tau = \tau_p < T\}} (L_{\tau_p} - \Phi_{\tau_p})^+ + \mathbb{1}_{\{\tau = \tau_c < \tau_p\}} (U_{\tau_c} - \Phi_{\tau_c}) + \mathbb{1}_{\{\tau = T\}} (\xi - \xi^b) \right) \end{aligned}$$

and we denote $\check{\mathcal{G}}_T^t = \{\tau \in \mathcal{G}_T^t; L_\tau \geq \Phi_\tau \text{ if } \tau < T\}$. For any $\tau_p \in \mathcal{G}_T^t$, the stopping time $\check{\tau}_p$, given by the formula

$$\check{\tau}_p = \mathbb{1}_{\{L_{\tau_p} \geq \Phi_{\tau_p}\}} \tau_p + \mathbb{1}_{\{L_{\tau_p} < \Phi_{\tau_p}\}} T,$$

belongs to $\check{\mathcal{G}}_T^t$. Moreover, since $\beta_t(\chi_t - \chi_t^b)$ is non-decreasing, $U \geq \Phi$ on $[\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$, and $\xi \geq \xi^b$, then the following inequalities hold, for any $\tau_p \in \mathcal{G}_T^t$ and any $\tau_c \in \bar{\mathcal{G}}_T^t$,

$$\psi_t(\check{\tau}_p, \tau_c) \geq \psi_t(\tau_p, \tau_c), \quad \check{\psi}_t(\check{\tau}_p, \tau_c) \geq \check{\psi}_t(\tau_p, \tau_c).$$

Since obviously $\check{\psi}_t(\tau_p, \tau_c) = \psi_t(\tau_p, \tau_c)$ for any $\tau_p \in \check{\mathcal{G}}_T^t$, we obtain

$$\begin{aligned} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\check{\psi}_t(\tau_p, \tau_c) | \mathcal{G}_t) &= \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \check{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}(\check{\psi}_t(\tau_p, \tau_c) | \mathcal{G}_t) \\ &= \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \check{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}(\psi_t(\tau_p, \tau_c) | \mathcal{G}_t) = \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\psi_t(\tau_p, \tau_c) | \mathcal{G}_t) \end{aligned}$$

and thus

$$\begin{aligned} \Psi_t &= \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}(\psi_t(\tau_p, \tau_c) | \mathcal{G}_t) \\ &= \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}(\check{\psi}_t(\tau_p, \tau_c) | \mathcal{G}_t) \geq 0. \end{aligned}$$

We conclude that $\Pi_t = \Phi_t + \Psi_t \geq \Phi_t$. \square

Note that Ψ does not need to be positive and Π does not need to be greater than Φ , in general.

5.4 Price Decomposition of a Reduced Convertible Bond

We shall now specialize our previous results to the case of a reduced convertible bond (hence, in particular, to the case of a convertible bond without call notice period). We thus postulate that the dividend process is of the form χ^{cb} defined in (16), that is

$$\beta_t \chi_t^{cb} = \sum_{0 \leq T_i \leq t, T_i < \tau_d} \beta_{T_i} c_i + \mathbb{1}_{\{0 \leq \tau_d \leq t\}} \beta_{\tau_d} R_{\tau_d}^{cb}, \quad t \in [0, T].$$

In order to provide the most pertinent price decomposition of the RB, we specify the bounded ES as the elementary security with dividend process

$$\beta_t \chi_t^b = \sum_{0 \leq T_i \leq t, T_i < \tau_d} \beta_{T_i} c_i + \mathbb{1}_{\{0 \leq \tau_d \leq t\}} \beta_{\tau_d} R_{\tau_d}^b, \quad t \in [0, T],$$

with the same coupon schedule $(T_i, c_i)_i$ as the RB, and with R^b and ξ^b as follows (see Section 4.1):

$$R_t^b = \bar{R}_t \bar{X}_t, \quad \xi^b = \bar{N} + A_T. \quad (29)$$

It is thus clear that

$$R_t^{cb} - R_t^b = (\kappa S_t - \bar{R}_t \bar{X}_t)^+ \geq 0, \quad \xi^{cb} - \xi^b = (\kappa S_T - \bar{N})^+ \geq 0.$$

So in this case the bounded ES corresponds to the defaultable bond with cash flows given by the expression

$$\begin{aligned} \beta_t \phi_t &= (\beta_T \chi_T^b - \beta_t \chi_t^b) + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi^b \\ &= \sum_{t < T_i \leq T, T_i < \tau_d} \beta_{T_i} c_i + \mathbb{1}_{\{t < \tau_d \leq T\}} \beta_{\tau_d} R_{\tau_d}^b + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi^b. \end{aligned}$$

This bond can be seen as the pure bond component of the RB (that is, the RB stripped of its optional clauses). Therefore, we shall call it the *bond embedded into the RB*, or simply the *embedded*

bond. Given also a probability measure $\mathbb{Q} \in \mathcal{M}$, the process $\Phi_t = \mathbb{E}_{\mathbb{Q}}(\phi_t | \mathcal{G}_t)$ is the \mathbb{Q} -price of the embedded bond.

Since the RB and the embedded bond have the same coupon schedule, the \mathbb{Q} -exchange CS is in fact a zero-coupon RB, with the cash flows given by the expression

$$\begin{aligned} \beta_t \psi_t(\tau_p, \tau_c) &= \mathbf{1}_{\{t < \tau_d \leq \tau\}} \beta_{\tau_d} (R_{\tau_d} - R_{\tau_d}^b) \\ &+ \mathbf{1}_{\{\tau_d > \tau\}} \beta_{\tau} \left(\mathbf{1}_{\{\tau = \tau_p < T\}} (L_{\tau_p} - \Phi_{\tau_p}) + \mathbf{1}_{\{\tau = \tau_c < \tau_p\}} (U_{\tau_c} - \Phi_{\tau_c}) + \mathbf{1}_{\{\tau = T\}} (\xi - \xi^b) \right). \end{aligned} \quad (30)$$

This particular \mathbb{Q} -exchange CS will be referred to as the *embedded game \mathbb{Q} -exchange option*. As for any \mathbb{Q} -exchange CS (see Section 5.2), the cash flows of the embedded game \mathbb{Q} -exchange option depend on the choice of a martingale measure \mathbb{Q} through the definition of the price process Φ of the embedded bond.

Theorem 5.1(i) or 5.1(ii) can be directly applied to this specification of an RB. Since an explicit representation for the price decomposition of an RB is easy to obtain, we do not report it here. The following result is worth stating, however.

Proposition 5.1 *Consider an RB and the embedded bond with R^b and ξ^b given by (29). Then, under the assumptions of Theorem 5.1(i) or 5.1(ii), we have that:*

- (i) *If the coupons c_i are non-negative, then $\Phi \geq 0$, and thus $\Pi \geq \Psi$,*
- (ii) *In the special case of a zero-coupon RB, then $\Psi \geq 0$, and thus $\Pi \geq \Phi$.*

Proof. Part (i) follows directly by an application of Corollary 5.1(i). In the special case of a zero-coupon RB, we have by (22)

$$\Phi_t \leq \bar{N} \leq \bar{C} \leq U_t^{cb}, \quad \tau_d \wedge \bar{\tau} \leq t < \tau_d \wedge T,$$

so that part (ii) is a consequence of Corollary 5.1(ii). \square

In general, Ψ does not need to be positive, and thus Π does not need to be greater than Φ . This is due, of course, to the game-theoretical features of the embedded option.

We can now formulate the definition of the spread and the implied volatility of an RB.

Definition 5.1 Let us consider an RB, under the assumptions of Theorem 5.1(i) or (ii). By the *RB spread*, we mean the credit spread consistent with the price Φ for the embedded bond. By an *RB implied volatility*, we mean any Black–Scholes volatility of the stock price process S , which is consistent with the price Ψ for the embedded game exchange option.

Remarks 5.2 (i) Note that it is not necessarily possible to map every possible arbitrage price process for the game exchange option to a well-defined and unique Black-Scholes implied volatility process.

(ii) The embedded game exchange option of an RB can be thought of as an equity option, but with a floating strike, equal at any date t to the current value Φ_t of the embedded bond. This clarifies the intuitive statements made at the beginning of this section, and confirms that the implied volatility of a CB (when properly defined) in Table 1 and the implied volatility for the closest listed option in Table 2 are in fact of a quite different nature.

6 Conclusions

As announced in the introduction, this note provides a theoretical underpinning for a more extensive research continued in Bielecki et al. [5, 6], where more specific market models are introduced, and more explicit valuation and hedging results are established. Some open problems remain, however.

Most notably, it is not clear whether it is possible to generalize Theorem 2.1 and Corollary 2.1 to the case of extended game options, such as the ones that arise naturally in the study of real-life convertible bond with positive call notice period. In the present paper and in the following ones, we develop an alternative approach based on a conjecture that the valuation of such a convertible bond can be done recursively.

One of the major problems arising in this context is that an arbitrage price of such a convertible bond upon call is not uniquely defined, unless we work in the framework of a complete market model. Consequently, it is not clear whether the pricing measures associated to arbitrage prices of the convertible bond and of the embedded puttable bonds have to be the same, even if one assumes that all the instruments involved are liquidly traded. In the same line of thinking, the decomposition of a convertible bond as the portfolio comprising a bond and an option can only be used under an ad hoc assumption that the same pricing measure is used to value the convertible bond, the embedded bond and the embedded game exchange option. Since is not a priori necessary when one deals with an incomplete market model, a further study of this issue is needed.

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